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1996 J. Phys. A: Math. Gen. 29 3129

(http://iopscience.iop.org/0305-4470/29/12/019)

J. Phys. A: Math. Gen. **29** (1996) 3129–3141. Printed in the UK

Spinorial representations of $SU(3)$ from a factored harmonic-oscillator equation

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Received 4 January 1996

Abstract. We construct a seven-dimensional Dirac equation for quarks by extending the Klein–Gordon equation to seven dimensions, introducing a harmonic-oscillator term in the extra three dimensions, and taking the 'square root' of the resulting equation. To facilitate solving the factored equation, we identify an $SU(3)$ algebra in the higher dimensions and use the eigenstates of its Cartan subalgebra as basis functions. We find that the eigenstates of the extended Dirac equation are indeed exact representations of $SU(3)$, one of which we identify with quarks. Indeed, a whole tower of $SU(3)$ Dirac states appears. For principal quantum number $\tilde{n} = 0$, we find a singlet of mass zero; for $\tilde{n} = 1$, we find two **3**s of mass $\sqrt{2}\tilde{\alpha}$, where $\tilde{\alpha}$ is an adjustable constant; for $\tilde{n} = 2$, we find two **6**s and two **3**-bars, all of mass $\sqrt{4}\tilde{\alpha}$; etc. Except for the ground state, all $SU(3)$ representations appear in pairs. It is not clear that the pairs can be identified with weak-isospin doublets.

1. Introduction

It is well known that the Schrödinger equation for a three-dimensional harmonic oscillator has solutions which are representations of an $SU(3)$ Lie algebra [1]. These solutions are ubiquitous in physics, e.g., ionic motion in crystals, independent-particle wavefunctions in the nuclear shell-model [2], etc.

$SU(3)$ is also the symmetry of quantum chromodynamics. However, here the symmetry shows up in the 'internal space' of the coloured quarks and gluons rather than in Minkowski space. We pose the following question. Might the $SU(3)$-symmetry result from a harmonic-oscillator (HO) 'potential' acting in three additional (flat) dimensions? It may not be outside the realm of possibility that a 'restraining force' could exist in higher dimensions. Rubakov and Shaposhnikov [3] have shown how the toy Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\sum_{A=0}^{4}(\partial_A\phi)(\partial^A\phi) + \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4}\lambda\phi^4 + \mathrm{i}\sum_{A=0}^{4}\overline{\psi}\gamma^A\partial_A\psi + g\overline{\psi}\phi\psi \qquad (1)$$

with metric $(+,-,-,-;-)$ can generate a classical solution $\phi \equiv \phi_c$ with a 'kink' or domain wall in the fifth dimension which can trap the Dirac particle. (The sign of $\frac{1}{2}m^2\phi^2$ in the Lagrangian generates the kink solution $\phi_c = (m/\sqrt{\lambda})\tanh(mx^{(4)}/\sqrt{2})$, where $x^{(4)}$ is the fifth dimension.)

In a relativistic field theory, of course, the field equation for a quark has to be the Dirac equation, not the Schrödinger equation. Unfortunately the ordinary Dirac equation for a HO potential does not generate $SU(3)$-symmetric wavefunctions, nor will an extended Dirac

0305-4470/96/123129+13$19.50

equation generate such $SU(3)$-symmetry in higher dimensions; the extended equation's solutions can be factored into an ordinary-space (Dirac) solution and a higher-dimensional-space solution all right, but the higher-dimensional wavefunction is only $SO(3)$-symmetric, essentially because the Dirac operator 'squares' the potential.

However, there is a relativistic equation that *does* generate exact $SU(3)$-symmetric wavefunctions if a three-dimensional HO term is inserted in it, and that is the Klein–Gordon equation. If a Klein–Gordon equation is extended to an extra three flat dimensions and a HO term inserted in the extra dimensions, then that equation will generate $SU(3)$-symmetric wavefunctions in those extra dimensions [4]. A tower of HO mass-eigenstates results, in the form of ascending $SU(3)$ $\mathbf{1}, \mathbf{3}, \mathbf{6}, \ldots$ multiplets.

Since factoring the ordinary Klein–Gordon equation yields a suitable equation for the QED electron (the Dirac equation), could factoring the extended Klein–Gordon equation yield a suitable equation for quark Dirac particles? That is, could factoring the extended Klein–Gordon equation yield an equation which generates ordinary Dirac wavefunctions in Minkowski-space times $SU(3)$-symmetric wavefunctions in the higher dimensions? Quarks could then be coupled to gluons in an $SU(3)$-invariant fashion, as required by QCD.

We will explore this hypothesis in the remainder of the paper.

2. Solutions to the factored higher-dimensional Klein–Gordon equation

The equation to be factored is the seven-dimensional Klein–Gordon equation

$$(\Box - \tilde{\Delta} + \tilde{\alpha}^4 \tilde{X}^2)\phi = 0 \tag{1}$$

where the Minkowski-space M^4 is spanned by the coordinates (x^0, x^1, x^2, x^3) and the higher-dimensional space $\tilde{R}^3$ is spanned by the coordinates $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. ($\tilde{R}^3$-quantities will be identified by subscripts rather than superscripts for convenience in what follows.) The d'Alembertian $\Box = \sum_{\mu=0}^{3} g^{\mu\nu}\partial^2/\partial x^\mu \partial x^\nu$ with $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, the Laplacian $\tilde{\Delta} = \sum_{\tilde{j}=1}^{3} \partial^2/\partial\tilde{x}_{\tilde{j}}^2$, and the HO term $\tilde{X}^2 \equiv \sum_{\tilde{j}=1}^{3} \tilde{x}_{\tilde{j}}^2$. The scale in $\tilde{R}^3$ is set by $\tilde{\alpha}$, an adjustable, real constant of dimension mass or inverse length†.

To factor (1), one may take advantage of the fact that the equation is separable; i.e. if one sets

$$\phi(x, \tilde{x}) = \phi(x)\tilde{\phi}(\tilde{x}) \tag{2}$$

then (1) separates into the equations

$$(\Box + M^2)\phi = 0 \tag{3}$$

and

$$(\tilde{\Delta} - \tilde{\alpha}^4 \tilde{X}^2 + M^2)\tilde{\phi} = 0 \tag{4}$$

where M^2 is the separation constant. Equations (3) and (4) can now be factored separately. Because the factors will commute, one can be taken from each equation and the two added to form the seven-dimensional Dirac equation.

Equation (3) is factored in the usual way; one can show that

$$\Delta + M^2 = (\mathrm{i}\,\nabla\!\!\!/ + M)(-\mathrm{i}\,\nabla\!\!\!/ + M) \tag{5}$$

† Symbols relating to M^4, $\tilde{R}^3$, or $M^4 \times \tilde{R}^3$ will appear: normally, with a tilde above, or in bold face type, respectively.

where $\nabla\!\!\!/ = \sum_{\mu=0}^{3} \gamma^\mu \partial/\partial x^\mu$, with the γ^μ being 4×4 matrices satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$. One can select one of the factors, say the right-most, to form the Dirac equation

$$(-\mathrm{i}\,\nabla\!\!\!/ + M)\psi = 0. \tag{6}$$

M is taken to be the mass of the particle.

Equation (4) in the higher dimensions cannot be factored so easily. We introduce a linear, first-order differential operator $\tilde{H}$ and pose the question, does

$$\tilde{\Delta} - \tilde{\alpha}^4\tilde{X}^2 + M^2 \stackrel{?}{=} (\tilde{H} + M)(-\tilde{H} + M) = -\tilde{H}^2 + M^2 \tag{7}$$

i.e. does

$$-\tilde{\Delta} + \tilde{\alpha}^4\tilde{X}^2 \stackrel{?}{=} \tilde{H}^2. \tag{8}$$

Three gamma-matrices would suffice to factor the left-hand side of (8) if the HO term were not present. As it is, another three gamma-matrices are required to eliminate the $\tilde{x}_{\tilde{j}}\partial/\partial\tilde{x}_{\tilde{k}}$ cross terms. Thus we try setting

$$\tilde{H} = -\mathrm{i}\sum_{\tilde{j}=1}^{3} \tilde{\gamma}_{\tilde{j}}\partial/\partial\tilde{x}_{\tilde{j}} + \tilde{\alpha}^2 \sum_{\tilde{j}=1}^{3} \tilde{\gamma}_{\tilde{j}+3}\tilde{x}_{\tilde{j}} \tag{9}$$

where

$$\tilde{\gamma}_{\tilde{j}}\tilde{\gamma}_{\tilde{k}} + \tilde{\gamma}_{\tilde{k}}\tilde{\gamma}_{\tilde{j}} = 2\delta_{\tilde{j}\tilde{k}} \qquad \tilde{j}, \tilde{k} = 1, 2, \ldots, 6. \tag{10}$$

Even with six gamma-matrices, (9) does not quite satisfy (8) because of the derivative operators in the cross terms. Instead (9) yields

$$\tilde{H}^2 = -\tilde{\Delta} + \tilde{\alpha}^4\tilde{X}^2 - \mathrm{i}\tilde{\alpha}^2 \sum_{\tilde{j}=1}^{3} \tilde{\gamma}_{\tilde{j}}\tilde{\gamma}_{\tilde{j}+3}. \tag{11}$$

However, choice (9) still turns out to be $SU(3)$-covariant, despite the extra term on the right-hand side. Therefore we take our Dirac field operator to be one of the factors in the centre of (7), say the right-most, and construct the candidate field equation

$$(\tilde{H} - M)\tilde{\psi}(\tilde{x}) = 0. \tag{12}$$

To guarantee that the mass M be real, we make $\tilde{H}$ Hermitian. This is accomplished by choosing six $\tilde{\gamma}_{\tilde{j}}$ which are Hermitian.

$\tilde{H}$ given by (9) is somewhat ungainly. Since it is the 'square root' of a HO operator, one suspects that it might be better to express (9) in terms of creation operators

$$\tilde{a}^\dagger_{\tilde{j}} = (\tilde{\alpha}\tilde{x}_{\tilde{j}} - \tilde{\alpha}^{-1}\partial/\partial\tilde{x}_{\tilde{j}})/\sqrt{2} \qquad \tilde{j} = 1, 2, 3 \tag{13}$$

and destruction operators

$$\tilde{a}_{\tilde{j}} = (\tilde{\alpha}\tilde{x}_{\tilde{j}} + \tilde{\alpha}^{-1}\partial/\partial\tilde{x}_{\tilde{j}})/\sqrt{2} \qquad \tilde{j} = 1, 2, 3 \tag{14}$$

satisfying

$$\tilde{a}_{\tilde{j}}\tilde{a}_{\tilde{k}} - \tilde{a}_{\tilde{k}}\tilde{a}_{\tilde{j}} = 0 \qquad \tilde{j}, \tilde{k} = 1, 2, 3 \tag{15}$$

$$\tilde{a}^\dagger_{\tilde{j}}\tilde{a}^\dagger_{\tilde{k}} - \tilde{a}^\dagger_{\tilde{k}}\tilde{a}^\dagger_{\tilde{j}} = 0 \qquad \tilde{j}, \tilde{k} = 1, 2, 3 \tag{16}$$

and

$$\tilde{a}_{\tilde{j}}\tilde{a}^\dagger_{\tilde{k}} - \tilde{a}^\dagger_{\tilde{k}}\tilde{a}_{\tilde{j}} = \delta_{\tilde{j}\tilde{k}} \qquad \tilde{j}, \tilde{k} = 1, 2, 3. \tag{17}$$

Indeed, if (9) is so expressed, then one finds that

$$\tilde{H} = \sqrt{2}\tilde{\alpha}\sum_{\tilde{j}=1}^{3}(\tilde{\theta}_{\tilde{j}}^{\dagger}\tilde{a}_{\tilde{j}} + \tilde{\theta}_{\tilde{j}}\tilde{a}_{\tilde{j}}^{\dagger}) \tag{18}$$

where

$$\tilde{\theta}_{\tilde{j}} = (\mathrm{i}\tilde{\gamma}_{\tilde{j}} + \tilde{\gamma}_{\tilde{j}+3})/2 \qquad \tilde{j} = 1, 2, 3 \tag{19}$$

and its Hermitian conjugate

$$\tilde{\theta}_{\tilde{j}}^{\dagger} = (-\mathrm{i}\tilde{\gamma}_{\tilde{j}} + \tilde{\gamma}_{\tilde{j}+3})/2 \qquad \tilde{j} = 1, 2, 3. \tag{20}$$

The six matrices $\tilde{\theta}_{\tilde{j}}^{\dagger}$ and $\tilde{\theta}_{\tilde{j}}$ turn out to be creation and annihilation operators, too, but of the anticommuting or Dirac variety; from (10) one finds that

$$\tilde{\theta}_{\tilde{j}}\tilde{\theta}_{\tilde{k}} + \tilde{\theta}_{\tilde{k}}\tilde{\theta}_{\tilde{j}} = 0 \qquad \tilde{j}, \tilde{k} = 1, 2, 3 \tag{21}$$

$$\tilde{\theta}_{\tilde{j}}^{\dagger}\tilde{\theta}_{\tilde{k}}^{\dagger} + \tilde{\theta}_{\tilde{k}}^{\dagger}\tilde{\theta}_{\tilde{j}}^{\dagger} = 0 \qquad \tilde{j}, \tilde{k} = 1, 2, 3 \tag{22}$$

and

$$\tilde{\theta}_{\tilde{j}}\tilde{\theta}_{\tilde{k}}^{\dagger} + \tilde{\theta}_{\tilde{k}}^{\dagger}\tilde{\theta}_{\tilde{j}} = \delta_{\tilde{j}\tilde{k}} \qquad \tilde{j}, \tilde{k} = 1, 2, 3. \tag{23}$$

We now construct the 'square root' of the seven-dimensional Klein–Gordon equation (1) by combining (6) and (12) (and (18)) to form

$$(-\mathrm{i}\ \nabla\!\!\!/ \otimes \tilde{1} + 1 \otimes \tilde{H})\boldsymbol{\psi} = 0 \tag{24}$$

here 1 and $\tilde{1}$ are unit matrices in M^4 and $\tilde{R}^3$, respectively, and $\boldsymbol{\psi} = \psi(x) \otimes \tilde{\psi}(\tilde{x})$. We will take (24) to be our candidate Dirac equation for quarks†.

We must now solve for the eigensolutions of $\tilde{H}$ in $\tilde{R}^3$. One suspects that these solutions will reveal $SU(3)$ symmetry because the solutions to the parent HO operator (4) display $SU(3)$ symmetry [4]. If $\tilde{H}$ is indeed $SU(3)$ symmetric, then there will be a set of $SU(3)$ generators which commute with it. We will look for such a set, as the eigenfunctions of the Cartan subalgebra can then be used as basis functions for $\tilde{\psi}$.

It is well known [5] that a $U(3)$ algebra can be created from the Bose creation and destruction operators $\tilde{a}_{\tilde{j}}^{\dagger}$ and $\tilde{a}_{\tilde{j}}$, namely, the algebra $\tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}}$, $\tilde{i}$, $\tilde{j} = 1, 2, 3$. These $\tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}}$ satisfy the $U(3)$ (and $SU(3)$) vector-multiplication rule

$$[\tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}}, \tilde{a}_{\tilde{k}}^{\dagger}\tilde{a}_{\tilde{l}}] = \delta_{\tilde{j}\tilde{k}}\tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{l}} - \delta_{\tilde{l}\tilde{i}}\tilde{a}_{\tilde{k}}^{\dagger}\tilde{a}_{\tilde{j}} \qquad \tilde{i}, \tilde{j}, \tilde{k}, \tilde{l} = 1, 2, 3. \tag{25}$$

The $\tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}}$ commute with the HO parent-operator (4) and account for the $SU(3)$-symmetry of its solutions. However, the $\tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}}$ do not commute with $\tilde{H}$. Rather,

$$[\tilde{H}, \tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}}] = \sqrt{2}\alpha(\tilde{\theta}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}} - \tilde{\theta}_{\tilde{j}}\tilde{a}_{\tilde{i}}^{\dagger}) \qquad \tilde{i}, \tilde{j} = 1, 2, 3. \tag{26}$$

† Technically the ten gamma-matrices in (24) are not Dirac matrices because the four matrices $\gamma^{\mu} \otimes \tilde{1}$ commute with the six matrices $1 \otimes \tilde{\gamma}_{\tilde{j}}$. However if we multiply (24) on the left by $1 \otimes \tilde{\gamma}_7$, where $\tilde{\gamma}_7 = \mathrm{i}\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_4\tilde{\gamma}_5\tilde{\gamma}_6$, then the resulting gamma-matrices in (24) do mutually anticommute and are true Dirac matrices; i.e. if we denote

$$\gamma^{\mu} \otimes \tilde{\gamma}_7 = \boldsymbol{\gamma}^{\boldsymbol{\mu}} \qquad \boldsymbol{\mu} = 0, 1, 2, 3 = \mu$$

$$1 \otimes \tilde{\gamma}_7\tilde{\gamma}_{\tilde{j}} = \boldsymbol{\gamma}^{\boldsymbol{\mu}} \qquad \boldsymbol{\mu} = 4, 5, \ldots, 9 = 3 + \tilde{j}$$

then

$$\boldsymbol{\gamma}^{\boldsymbol{\mu}}\boldsymbol{\gamma}^{\boldsymbol{\nu}} + \boldsymbol{\gamma}^{\boldsymbol{\nu}}\boldsymbol{\gamma}^{\boldsymbol{\mu}} = 2\boldsymbol{g}^{\boldsymbol{\mu\nu}} \qquad \boldsymbol{\mu}, \boldsymbol{\nu} = 0, 1, 2, \ldots, 9$$

where $\boldsymbol{g} = \mathrm{diag}(1, -1, -1, \ldots, -1)$. The solutions $\boldsymbol{\psi}$ of (24) are unaffected if the equation is multiplied on the left by $1 \otimes \tilde{\gamma}_7$, so we shall continue to refer to (24) as the candidate Dirac equation for quarks.

Thus the $\tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}}$ cannot be generators of any $SU(3)$ symmetry that solutions to (12) might embody.

Another $U(3)$-algebra can be constructed from the Dirac creation and destruction operators $\tilde{\theta}_{\tilde{j}}^{\dagger}$ and $\tilde{\theta}_{\tilde{j}}$, that algebra being the $\tilde{\theta}_{\tilde{i}}^{\dagger}\tilde{\theta}_{\tilde{j}}$, $\tilde{i}, \tilde{j} = 1, 2, 3$. From (21)–(23), one can show that this algebra satisfies the $U(3)$ vector-multiplication rule

$$[\tilde{\theta}_{\tilde{i}}^{\dagger}\tilde{\theta}_{\tilde{j}}, \tilde{\theta}_{\tilde{k}}^{\dagger}\tilde{\theta}_{\tilde{l}}] = \delta_{\tilde{j}\tilde{k}}\tilde{\theta}_{\tilde{i}}^{\dagger}\tilde{\theta}_{\tilde{l}} - \delta_{\tilde{l}\tilde{i}}\tilde{\theta}_{\tilde{k}}^{\dagger}\tilde{\theta}_{\tilde{j}} \qquad \tilde{i}, \tilde{j}, \tilde{k}, \tilde{l} = 1, 2, 3. \tag{27}$$

However, this algebra does not commute with $\tilde{H}$ either. Instead

$$[\tilde{H}, \tilde{\theta}_{\tilde{i}}^{\dagger}\tilde{\theta}_{\tilde{j}}] = -\sqrt{2}\alpha(\tilde{\theta}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}} - \tilde{\theta}_{\tilde{j}}\tilde{a}_{\tilde{i}}^{\dagger}) \qquad \tilde{i}, \tilde{j} = 1, 2, 3. \tag{28}$$

But one can see from (26) and (28) that $\tilde{H}$ does commute with the sum

$$\tilde{E}_{\tilde{i}\tilde{j}} = \tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}} + \tilde{\theta}_{\tilde{i}}^{\dagger}\tilde{\theta}_{\tilde{j}} \qquad \tilde{i}, \tilde{j} = 1, 2, 3 \tag{29}$$

viz.,

$$[\tilde{H}, \tilde{E}_{\tilde{i}\tilde{j}}] = 0 \qquad \tilde{i}, \tilde{j} = 1, 2, 3. \tag{30}$$

Furthermore, the $\tilde{E}_{\tilde{i}\tilde{j}}$, being the sum of the $\tilde{a}_{\tilde{i}}^{\dagger}\tilde{a}_{\tilde{j}}$ and the $\tilde{\theta}_{\tilde{i}}^{\dagger}\tilde{\theta}_{\tilde{j}}$, also satisfy a $U(3)$ vector-multiplication rule:

$$[\tilde{E}_{\tilde{i}\tilde{j}}, \tilde{E}_{\tilde{k}\tilde{l}}] = \delta_{\tilde{j}\tilde{k}}\tilde{E}_{\tilde{i}\tilde{l}} - \delta_{\tilde{l}\tilde{i}}\tilde{E}_{\tilde{k}\tilde{j}} \qquad \tilde{i}, \tilde{j}, \tilde{k}, \tilde{l} = 1, 2, 3. \tag{31}$$

These generators are apparently the sought-for $U(3)$ algebra. They guarantee that the eigenfunctions of $\tilde{H}$ will be $U(3)$ (and $SU(3)$) symmetric†.

We can now solve for the eigenfunctions of $\tilde{H}$, advantageously using representations of the $\tilde{E}_{\tilde{i}\tilde{j}}$ as basis functions. As noted earlier, these representations will be eigenfunctions of the Cartan subalgebra ($\tilde{E}_{11}$, $\tilde{E}_{22}$, $\tilde{E}_{33}$). We will solve for the eigenfunctions by first constructing explicit representations of the six $\tilde{\theta}_{\tilde{i}}$ and $\tilde{\theta}_{\tilde{i}}^{\dagger}$, and then constructing $\tilde{\theta}_1^{\dagger}\tilde{\theta}_1$, $\tilde{\theta}_2^{\dagger}\tilde{\theta}_2$, and $\tilde{\theta}_3^{\dagger}\tilde{\theta}_3$.

Recall that six gamma-matrices are required to construct the $\tilde{\theta}_{\tilde{i}}$ and $\tilde{\theta}_{\tilde{i}}^{\dagger}$. Wilczek and Zee [6] remind us that representing six (or seven) gamma-matrices requires 8×8 matrices, and they provide a prescription for constructing them. Starting with their rule, we found it possible to order the gamma-matrices so that each of the six $\tilde{\theta}_{\tilde{i}}$ and $\tilde{\theta}_{\tilde{i}}^{\dagger}$ has just four elements of 1 or -1; one set is

$$\tilde{\theta}_1 = \begin{pmatrix} 0 & -e_{21} & 0 & 0 \\ e_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{21} \\ 0 & 0 & -e_{21} & 0 \end{pmatrix} \qquad \tilde{\theta}_1^{\dagger} = \begin{pmatrix} 0 & e_{12} & 0 & 0 \\ -e_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_{12} \\ 0 & 0 & e_{12} & 0 \end{pmatrix}$$

† We were led to the generators $\tilde{E}_{\tilde{i}\tilde{j}}$ from a study in $\tilde{R}^4$ of the Dirac-equation $SO(4)$ generators

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \qquad \mu, \nu = 0, 1, 2, 3$$

where

$$L_{\mu\nu} = -\mathrm{i}(x_\mu \partial/\partial x_\nu - x_\nu \partial/\partial x_\mu) = -\mathrm{i}(a_\mu^{\dagger}a_\nu - a_\nu^{\dagger}a_\mu)$$

and

$$S_{\mu\nu} = -(\mathrm{i}/4)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu);$$

thus one can write

$$J_{\mu\nu} = -\mathrm{i}(a_\mu^{\dagger}a_\nu + \tfrac{1}{4}\gamma_\mu\gamma_\nu) + \mathrm{i}(a_\nu^{\dagger}a_\mu + \tfrac{1}{4}\gamma_\nu\gamma_\mu).$$

We found that $a_\mu^{\dagger}a_\nu + \frac{1}{4}\gamma_\mu\gamma_\nu$ is not an $SU(4)$ generator (of course), but that $a_\mu^{\dagger}a_\nu + \theta_\mu^{\dagger}\theta_\nu$ is.

$$\tilde\theta_2 = \begin{pmatrix} 0 & e_{22} & 0 & 0 \\ e_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_{22} \\ 0 & 0 & -e_{11} & 0 \end{pmatrix} \qquad \tilde\theta_2^\dagger = \begin{pmatrix} 0 & e_{11} & 0 & 0 \\ e_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_{11} \\ 0 & 0 & -e_{22} & 0 \end{pmatrix}$$

and

$$\tilde\theta_3 = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \tilde\theta_3^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \tag{32}$$

where

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{33}$$

With this choice of the $\tilde\theta_{\tilde i}$ and $\tilde\theta_{\tilde i}^\dagger$, the Cartan subalgebra takes the form

$$\tilde E_{11} = \tilde a_1^\dagger \tilde a_1 + \begin{pmatrix} e_{11} & 0 & 0 & 0 \\ 0 & e_{11} & 0 & 0 \\ 0 & 0 & e_{11} & 0 \\ 0 & 0 & 0 & e_{11} \end{pmatrix} \tag{34}$$

$$\tilde E_{22} = \tilde a_2^\dagger \tilde a_2 + \begin{pmatrix} e_{11} & 0 & 0 & 0 \\ 0 & e_{22} & 0 & 0 \\ 0 & 0 & e_{11} & 0 \\ 0 & 0 & 0 & e_{22} \end{pmatrix} \tag{35}$$

and

$$\tilde E_{33} = \tilde a_3^\dagger \tilde a_3 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}. \tag{36}$$

To solve for the eigenfunctions of the Cartan subalgebra, it is convenient to recall the eigenfunctions of the operators $\tilde a_{\tilde i}^\dagger \tilde a_{\tilde i}$, $\tilde i = 1, 2, 3$. These eigenfunctions satisfy

$$\tilde a_{\tilde i}^\dagger \tilde a_{\tilde i} \tilde\phi_{\tilde n_{\tilde i}}(\tilde\alpha \tilde x_{\tilde i}) = \tilde n_{\tilde i} \tilde\phi_{\tilde n_{\tilde i}}(\tilde\alpha \tilde x_{\tilde i}) \qquad \tilde i = 1, 2, 3,\ \ \tilde n_{\tilde i} = 0, 1, 2, \ldots \tag{37}$$

(no sum on $\tilde i$), where

$$\tilde\phi_{\tilde n_{\tilde i}}(\tilde\alpha \tilde x_{\tilde i}) = \text{constant } H_{\tilde n_{\tilde i}}(\tilde\alpha \tilde x_{\tilde i}) \exp(-\tfrac{1}{2}\tilde\alpha^2 \tilde x_{\tilde i}^2) \tag{38}$$

with $H_{\tilde n_{\tilde i}}$ a Hermite polynomial of degree $\tilde n_{\tilde i}$. Thus an eigenfunction of the trio of operators $\tilde a_1^\dagger \tilde a_1$, $\tilde a_2^\dagger \tilde a_2$, and $\tilde a_3^\dagger \tilde a_3$ is the product

$$\prod_{\tilde i = 1}^{3} \tilde\phi_{\tilde n_{\tilde i}}(\tilde\alpha \tilde x_{\tilde i}) \equiv \tilde\phi_{\tilde n_1, \tilde n_2, \tilde n_3}. \tag{39}$$

Now consider the eigenfunctions of the Cartan subalgebra ($\tilde E_{11}$, $\tilde E_{22}$, $\tilde E_{33}$). Each of these generators is a diagonal 8×8 matrix with elements either $\tilde a_{\tilde i}^\dagger \tilde a_{\tilde i}$ or $\tilde a_{\tilde i}^\dagger \tilde a_{\tilde i} + 1$. Therefore the three $\tilde{E}_{\tilde{i}\tilde{i}}$ have eight-component eigenfunctions $\tilde{\Phi}$ which require only a single non-zero component $\tilde{\phi}_{\tilde{n}'_1}\tilde{\phi}_{\tilde{n}'_2}\tilde{\phi}_{\tilde{n}'_3}$, $\tilde{n}'_{\tilde{i}} = 0, 1, 2, \ldots$, $\tilde{i} = 1, 2, 3$ to satisfy

$$\tilde{E}_{11}\tilde{\Phi} = \tilde{n}_1\tilde{\Phi} \qquad \tilde{E}_{22}\tilde{\Phi} = \tilde{n}_2\tilde{\Phi} \qquad \tilde{E}_{33}\tilde{\Phi} = \tilde{n}_3\tilde{\Phi}; \tag{40}$$

here each $\tilde{n}_{\tilde{i}} = \tilde{n}'_{\tilde{i}}$ or $\tilde{n}'_{\tilde{i}} + 1$ depending on whether the corresponding element in $\tilde{E}_{\tilde{i}\tilde{i}}$ is $\tilde{a}^{\dagger}_{\tilde{i}}\tilde{a}_{\tilde{i}}$ or $\tilde{a}^{\dagger}_{\tilde{i}}\tilde{a}_{\tilde{i}} + 1$. Thus the $\tilde{n}_{\tilde{i}}$ are non-negative integers.

For every triplet of non-negative integers $(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$, there are (up to) eight linearly independent solutions $\tilde{\Phi}$, namely

$$\begin{aligned}
&\tilde{\Phi}^{(1)}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} = (\tilde{\phi}_{\tilde{n}_1-1,\tilde{n}_2-1,\tilde{n}_3}, 0, 0, 0, 0, 0, 0, 0)^T \\
&[3pt]\tilde{\Phi}^{(2)}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} = (0, \tilde{\phi}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3}, 0, 0, 0, 0, 0, 0)^T \\
&[3pt]\tilde{\Phi}^{(3)}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} = (0, 0, \tilde{\phi}_{\tilde{n}_1-1,\tilde{n}_2,\tilde{n}_3}, 0, 0, 0, 0, 0)^T \\
&[3pt]\tilde{\Phi}^{(4)}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} = (0, 0, 0, \tilde{\phi}_{\tilde{n}_1,\tilde{n}_2-1,\tilde{n}_3}, 0, 0, 0, 0)^T \\
&[3pt]\tilde{\Phi}^{(5)}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} = (0, 0, 0, 0, \tilde{\phi}_{\tilde{n}_1-1,\tilde{n}_2-1,\tilde{n}_3-1}, 0, 0, 0)^T \\
&[3pt]\tilde{\Phi}^{(6)}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} = (0, 0, 0, 0, 0, \tilde{\phi}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3-1}, 0, 0)^T \\
&[3pt]\tilde{\Phi}^{(7)}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} = (0, 0, 0, 0, 0, 0, \tilde{\phi}_{\tilde{n}_1-1,\tilde{n}_2,\tilde{n}_3-1}, 0)^T \\
&[3pt]\tilde{\Phi}^{(8)}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} = (0, 0, 0, 0, 0, 0, 0, \tilde{\phi}_{\tilde{n}_1,\tilde{n}_2-1,\tilde{n}_3-1})^T.
\end{aligned} \tag{41}$$

(T denotes transpose, employed to save space.) Note that the indices of each element $\tilde{\phi}_{\tilde{n}'_1,\tilde{n}'_2,\tilde{n}'_3}$ must be $\geqslant 0$.

These solutions may be conveniently classified according to their principal quantum number $\tilde{n} \equiv \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3$. This is the eigenvalue of the Casimir operator

$$\tilde{E}_{11} + \tilde{E}_{22} + \tilde{E}_{33} \equiv \tilde{C}_1 = \tilde{N}. \tag{42}$$

For $\tilde{n} = 0$ there is just the one eigenfunction, $\tilde{\Phi}^{(2)}_{000}$. This is a singlet $SU(3)$ irreducible representation which we will denote $\tilde{\Phi}_0$. (We know that it is a singlet because every shift operator acting on it yields zero, viz., $\tilde{E}_{\tilde{i}\tilde{j}}\tilde{\Phi}_0 = 0$; $\tilde{i}, \tilde{j} = 1, 2, 3, \quad \tilde{i} \neq \tilde{j}$.)

For $\tilde{n} = 1$, there are six eigenfunctions: $\tilde{\Phi}^{(2)}_{100}$, $\tilde{\Phi}^{(2)}_{010}$, $\tilde{\Phi}^{(2)}_{001}$, and $\tilde{\Phi}^{(3)}_{000}$, $\tilde{\Phi}^{(4)}_{000}$, $\tilde{\Phi}^{(6)}_{000}$. To determine what irreps these make up, one may start with any eigenfunction and construct the remaining members of the irrep using the shift operators $\tilde{E}_{21}$, $\tilde{E}_{32}$ and $\tilde{E}_{13}$. In this way one finds two triplet irreps which we will denote† $\tilde{\Phi}^{(a1)}_1$ and $\tilde{\Phi}^{(a2)}_1$.

Their eigenfunctions are listed in figure 1 next to plots of their $SU(3)$ eigenvalues $(\tilde{m}_1, \tilde{m}_2)$, where $\tilde{m}_1 = \frac{1}{2}(\tilde{n}_1 - \tilde{n}_2)$ and $\tilde{m}_2 = (\tilde{n}_1 + \tilde{n}_2 - 2\tilde{n}_3)/\sqrt{12}$. These triplets may be distinguished by the eigenvalues of the operator

$$\tilde{A} = \tilde{a}^{\dagger}_1\tilde{a}_1 + \tilde{a}^{\dagger}_2\tilde{a}_2 + \tilde{a}^{\dagger}_3\tilde{a}_3; \tag{43}$$

$\tilde{A}\tilde{\Phi}_{\tilde{n}} = \tilde{A}'\tilde{\Phi}_{\tilde{n}}$ with $\tilde{A}' = 1$ for $\tilde{\Phi}^{(a1)}_1$ and 0 for $\tilde{\Phi}^{(a2)}_1$.

For $\tilde{n} = 2$ there are 18 eigenfunctions of the Cartan subalgebra. One can show that these comprise two **6**s which we will denote $\tilde{\Phi}^{(a1)}_2$ and $\tilde{\Phi}^{(a2)}_2$, and two $\bar{\mathbf{3}}$s which we will denote $\tilde{\Phi}^{(b1)}_2$ and $\tilde{\Phi}^{(b2)}_2$. The **6**s have eigenvalues $\tilde{A}' = 2$ and 1, respectively, and the $\bar{\mathbf{3}}$s have eigenvalues $\tilde{A}' = 1$ and 0, respectively. Weight diagrams of these multiplets are sketched in figure 2.

† When referring to an entire multiplet, we will replace the triplet subscripts $(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ by $\tilde{n}$.

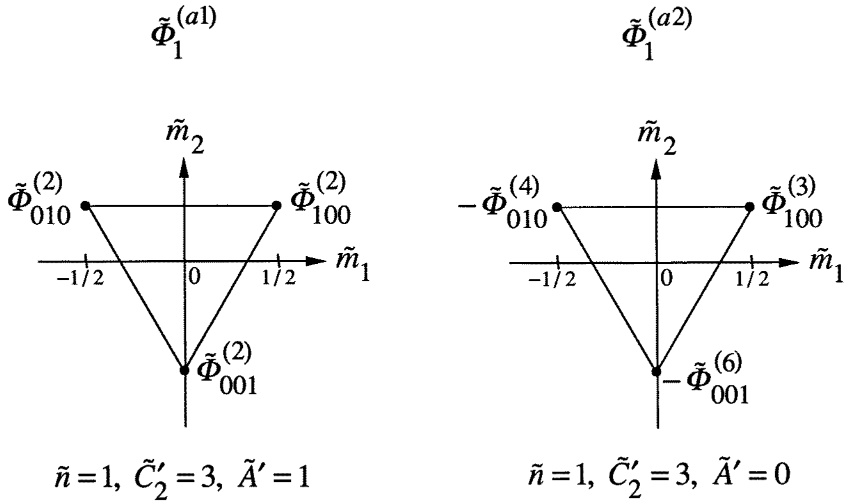

Figure 1. $SU(3)$ weight diagrams of the two $\tilde{n}=1$ irreducible representations of the generators $\tilde{E}_{\tilde{i}\tilde{j}}$, $\tilde{i},\tilde{j}=1,2,3$; basis functions listed at weight positions; note signs. Eigenvalues for each irrep listed underneath.

For $\tilde{n}=3$ and greater there are always six multiplets; they always appear in pairs, and the multiplets within each pair always have eigenvalues $\tilde{A}'$ which differ by one unit. Multiplets through $\tilde{n}=3$ are sketched in figure 2.

It is now a simple matter to solve for the eigenfunctions of $\tilde{H}\tilde{\psi}=M\tilde{\psi}$. One finds by direct calculation that

$$\tilde{H}\tilde{\Phi}_0=0. \tag{44}$$

Thus the lowest-lying eigenstate of $\tilde{H}$ is

$$\tilde{\Phi}_0\equiv\tilde{\psi}_0 \tag{45}$$

with eigenmass $M_0=0$.

Next one finds that

$$\tilde{H}\tilde{\Phi}_1^{(a1)}=-\sqrt{2}\tilde{\alpha}\tilde{\Phi}_1^{(a2)} \tag{46}$$

and

$$\tilde{H}\tilde{\Phi}_1^{(a2)}=-\sqrt{2}\tilde{\alpha}\tilde{\Phi}_1^{(a1)}. \tag{47}$$

Thus the next lowest eigenstates of $\tilde{H}$ are

$$(\tilde{\Phi}_1^{(a1)}-\tilde{\Phi}_1^{(a2)})/\sqrt{2}\equiv\tilde{\psi}_1^{(a+)} \tag{48}$$

and

$$(\tilde{\Phi}_1^{(a1)}+\tilde{\Phi}_1^{(a2)})/\sqrt{2}\equiv\tilde{\psi}_1^{(a-)} \tag{49}$$

with eigenmasses $M_1=\sqrt{2}\tilde{\alpha}$ and $-\sqrt{2}\tilde{\alpha}$, respectively.

In the case of the $\tilde{n}=2$ sextets one finds that

$$\tilde{H}\tilde{\Phi}_2^{(a1)}=-\sqrt{4}\tilde{\alpha}\tilde{\Phi}_2^{(a2)} \tag{50}$$

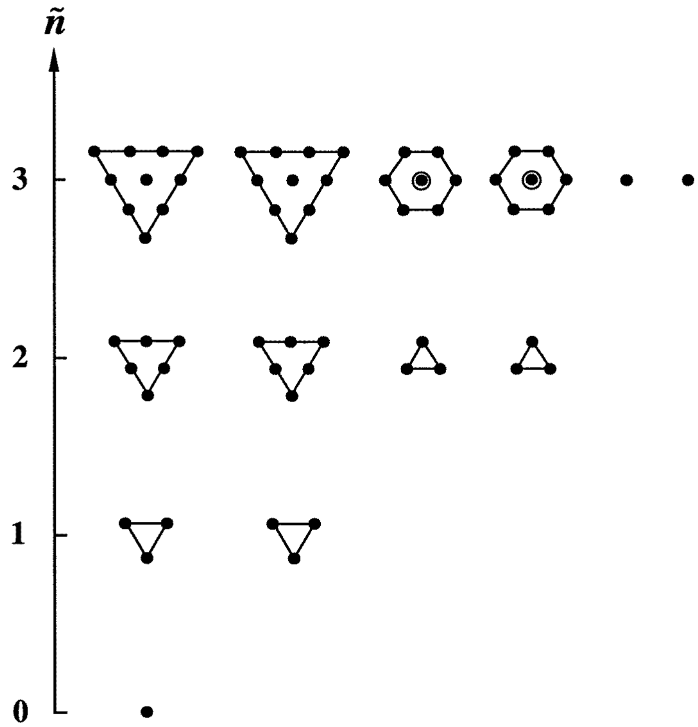

Figure 2. Weight diagrams of $\tilde{n} = 0$–3 irreps of the $SU(3)$ generators $\tilde{E}_{\tilde{i}\tilde{j}}$. These are also the weight diagrams of the $\tilde{n} = 0$–3 eigenfunctions of the mass operator $\tilde{H}$.

and

$$\tilde{H}\tilde{\Phi}_2^{(a2)} = -\sqrt{4}\tilde{\alpha}\tilde{\Phi}_2^{(a1)} \tag{51}$$

so the corresponding eigenfunctions of $\tilde{H}$ are

$$(\tilde{\Phi}_2^{(a1)} - \tilde{\Phi}_2^{(a2)})/\sqrt{2} \equiv \tilde{\psi}_2^{(a+)} \tag{52}$$

and

$$(\tilde{\Phi}_2^{(a1)} + \tilde{\Phi}_2^{(a2)})/\sqrt{2} \equiv \tilde{\psi}_2^{(a-)} \tag{53}$$

with eigenmasses $M_2 = \sqrt{4}\tilde{\alpha}$ and $-\sqrt{4}\tilde{\alpha}$, respectively. Similarly, in the case of the $\tilde{n} = 2$ triplets one finds that

$$\tilde{H}\tilde{\Phi}_2^{(b1)} = -\sqrt{4}\tilde{\alpha}\tilde{\Phi}_2^{(b2)} \tag{54}$$

and

$$\tilde{H}\tilde{\Phi}_2^{(b2)} = -\sqrt{4}\tilde{\alpha}\tilde{\Phi}_2^{(b1)} \tag{55}$$

so the corresponding eigenfunctions of $\tilde{H}$ are

$$(\tilde{\Phi}_2^{(b1)} - \tilde{\Phi}_2^{(b2)})/\sqrt{2} \equiv \tilde{\psi}_2^{(b+)} \tag{56}$$

and

$$(\tilde{\Phi}_2^{(b1)} + \tilde{\Phi}_2^{(b2)})/\sqrt{2} \equiv \tilde{\psi}_2^{(b-)} \tag{57}$$

again with eigenmasses $M_2 = \sqrt{4}\tilde{\alpha}$ and $-\sqrt{4}\tilde{\alpha}$, respectively.

Thus $\tilde{H}$ not only generates $SU(3)$ multiplets, but an ascending mass-spectrum of them. We might have anticipated this from the properties of the parent, equation (4).

Note that $\tilde{H}$ acts to term one $\tilde{n} = 2$ sextet into just the other sextet, or one $\tilde{n} = 2$ triplet into just the other triplet. Why is this?

(i) $\tilde{H}$ commutes with the (second) Casimir operator

$$\sum_{\tilde{i},\tilde{j}=1}^{3} \tilde{E}_{\tilde{i}\tilde{j}} \tilde{E}_{\tilde{j}\tilde{i}} \equiv \tilde{C}_2 \tag{58}$$

so it cannot change the eigenvalues of $\tilde{C}_2$ (which we will denote $\tilde{C}_2'$); the sextets have one common eigenvalue $\tilde{C}_2'$, the triplets another;

$$\tilde{C}_2 \tilde{\Phi}_2^{(ai)} = 8\tilde{\Phi}_2^{(ai)} \qquad i = 1, 2 \tag{59}$$

and

$$\tilde{C}_2 \tilde{\Phi}_2^{(bi)} = 4\tilde{\Phi}_2^{(bi)} \qquad i = 1, 2. \tag{60}$$

(ii) $\tilde{H}$ is odd in $\tilde{a}_{\tilde{j}}$ and $\tilde{a}_{\tilde{j}}^{\dagger}$, so it changes the value of $\tilde{A}'$ by one unit.

For $\tilde{n} = 3$ multiplets, where three pairs of like multiplets occur, $\tilde{H}$ acts on one $SU(3)$ multiplet and just turns it into its sister multiplet because each pair shares a common eigenvalue $\tilde{C}_2'$ and the values of the three pairs are distinct. Thus the $\tilde{n} = 3$ eigenfunctions of $\tilde{H}$ are

$$(\tilde{\Phi}_3^{(m1)} \mp \tilde{\Phi}_3^{(m2)})/\sqrt{2} \equiv \tilde{\psi}_3^{(m\pm)} \qquad m = a, b, c \tag{61}$$

with eigenmasses $M_3 = \pm\sqrt{6}\tilde{\alpha}$.

Similar eigenfunctions $\tilde{\psi}_{\tilde{n}}^{(a\pm)}$, $\tilde{\psi}_{\tilde{n}}^{(b\pm)}$ and $\tilde{\psi}_{\tilde{n}}^{(c\pm)}$ occur for $\tilde{n} > 3$, with eigenmasses $M_{\tilde{n}} = \pm\sqrt{2\tilde{n}}\tilde{\alpha}$.

We can now determine the solutions to the candidate Dirac field equation

$$(-\mathrm{i}\,\nabla\!\!\!/ \otimes \tilde{1} + 1 \otimes \tilde{H})\boldsymbol{\psi} = 0. \tag{62}$$

(See footnote below (24).) The solutions are simple direct products $\boldsymbol{\psi} = \psi(x) \otimes \tilde{\psi}(\tilde{x})$, where $\psi(x)$ satisfies the M^4-equation

$$(-\mathrm{i}\,\nabla\!\!\!/ + M)\psi = 0 \tag{63}$$

and $\tilde{\psi}(\tilde{x})$ satisfies the $\tilde{R}^3$-equation

$$(\tilde{H} - M)\tilde{\psi}(\tilde{x}) = 0. \tag{64}$$

If the mass M of the multiplet is positive, as in the case for $\tilde{\psi}(\tilde{x}) = \tilde{\psi}_{\tilde{n}}^{(m+)}$, $m = a, b, c$, then the Minkowski-space wavefunction ψ satisfies the equation

$$\left(-\mathrm{i}\,\nabla\!\!\!/ + \sqrt{2\tilde{n}}\tilde{\alpha}\right)\psi_{\tilde{n}}(x) = 0. \tag{65}$$

(Indices denoting momentum $\vec{p}$ and mechanical spin π in Minkowski space are suppressed, i.e., $\psi_{\vec{p},\pi,M_{\tilde{n}}}(x) \equiv \psi_{\tilde{n}}(x)$.) If the mass M of the multiplet is negative, as is the case for $\tilde{\psi}(\tilde{x}) = \tilde{\psi}_{\tilde{n}}^{(m-)}$, $m = a, b, c$, then the corresponding Minkowski-space wave equation can be multiplied on the left by $\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ to yield the positive-mass equation

$$\left(-\mathrm{i}\,\nabla\!\!\!/ + \sqrt{2\tilde{n}}\tilde{\alpha}\right)\gamma^5\psi_{\tilde{n}}(x) = 0. \tag{66}$$

Thus we arrive at the pair of solutions

$$\psi_{\tilde{n}}(x) \otimes \tilde{\psi}_{\tilde{n}}^{(m+)}(\tilde{x}) \equiv \boldsymbol{\psi}_{\tilde{\boldsymbol{n}}}^{(m+)}(\boldsymbol{x}) \tag{67}$$

and

$$\gamma^5 \psi_{\tilde{n}}(x) \otimes \tilde{\psi}_{\tilde{n}}^{(m-)}(\tilde{x}) \equiv \boldsymbol{\psi}_{\tilde{\boldsymbol{n}}}^{(m-)}(\boldsymbol{x}) \tag{68}$$

with $M_{\tilde{n}} = \sqrt{2\tilde{n}}\tilde{\alpha}$.

The full complement of solutions consists of the ground state $\psi_0 \otimes \tilde{\psi}_0 \equiv \boldsymbol{\psi}_0$, plus ascending pairs $\boldsymbol{\psi}_{\tilde{\boldsymbol{n}}}^{(m\pm)}$ for $\tilde{n} = 1, 2, 3, \ldots$, with $m = a$ for $\tilde{n} = 1$, $m = a, b$ for $\tilde{n} = 2$, and $m = a, b, c$ for $\tilde{n} \geqslant 3$. The weight diagrams for eigenstates with low $\tilde{n}$ are shown in figure 2.

3. Discussion

In order to generate Dirac eigenstates augmented by QCD colour quantum numbers, we have proposed the field equation (24), which is essentially the 'square root' of a Klein–Gordon equation in seven flat dimensions with a HO term in the extra three dimensions. Dirac states with quark colour quantum numbers do indeed appear, namely the two solutions denoted $\boldsymbol{\psi}_1^{(a\pm)}$ in section 2. These states can be coupled to eight coloured gluons $W_\mu^{\tilde{i}\tilde{j}}$ to yield the $SU(3)$-invariant interaction Lagrangian

$$\mathcal{L}^{\text{int}} = \frac{g}{\sqrt{2}} \sum_{\tilde{i}=1}^{3} \sum_{\tilde{j}=1}^{3} \int\int\int \overline{\boldsymbol{\psi}} \gamma^\mu (\tilde{E}_{\tilde{i}\tilde{j}} - \tfrac{1}{3}\tilde{N}) W_\mu^{\tilde{i}\tilde{j}} \boldsymbol{\psi} \, \mathrm{d}^3 x \tag{69}$$

($\tilde{N}$ is defined by (42)). If states $\boldsymbol{\psi}$ in (69) are limited to quark triplets, then this Lagrangian is equivalent to the standard QCD interaction Lagrangian

$$\mathcal{L}_{\text{QCD}}^{\text{int}} = \frac{g}{2} \sum_{c,d=1}^{3} \overline{\psi}_c \gamma^\mu \lambda_{cd}^k G_\mu^k \psi_d \tag{70}$$

(see, e.g., [7]), where

$$G_\mu^3 = (W_\mu^{11} - W_\mu^{22})/\sqrt{2} \tag{71}$$

$$G_\mu^8 = (W_\mu^{11} + W_\mu^{22} - 2W_\mu^{33})/\sqrt{6} \tag{72}$$

and

$$G_\mu^1 = (W_\mu^{12} + W_\mu^{21})/\sqrt{2} \tag{73}$$

$$G_\mu^2 = \mathrm{i}(W_\mu^{12} - W_\mu^{21})/\sqrt{2} \tag{74}$$

etc.

Equation (24) generates a surprisingly rich spectrum of states, especially considering that all of the eigenfunctions are real. Particularly interesting is the feature that, except for the ground state, all of the multiplets appear in pairs. Thus it is natural to ask whether these pairs can be identified with another symmetry of standard-model Dirac particles, namely, weak isospin.

If these pairs are indeed *bona fide* weak isospin pairs, then there exist operators, say $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$, $\tilde{\Sigma}_3$, and $\tilde{I}$, which, when operating on one member of any doublet, (i) only turn it into a linear combination of the same doublet states, and (ii) obey the vector-multiplication rules of $SU(2)$, namely

$$[\tilde{\Sigma}_{\tilde{i}}, \tilde{\Sigma}_{\tilde{j}}] = 2\mathrm{i}\tilde{\Sigma}_{\tilde{k}} \qquad \tilde{i}, \tilde{j}, \tilde{k} = 1, 2, 3 \text{ cyclic.} \tag{75}$$

Condition (i) is satisfied if the operator commutes with the $1 \otimes \tilde{E}_{\tilde{i}\tilde{j}}$, $\tilde{i}, \tilde{j} = 1, 2, 3$. We have found four such operators, viz.,

$$\gamma^5 \otimes \sum_{\tilde{i}=1}^{3} (\tilde{\theta}_{\tilde{i}}^\dagger \tilde{a}_{\tilde{i}} + \tilde{\theta}_{\tilde{i}} \tilde{a}_{\tilde{i}}^\dagger) \equiv \tilde{\Sigma}_1' \ \left(= \tilde{H}/\sqrt{2}\tilde{\alpha}\right) \tag{76}$$

$$\mathrm{i}\gamma^5 \otimes \sum_{\tilde{i}=1}^{3} (\tilde{\theta}_{\tilde{i}}^\dagger \tilde{a}_{\tilde{i}} - \tilde{\theta}_{\tilde{i}} \tilde{a}_{\tilde{i}}^\dagger) \equiv \tilde{\Sigma}_2' \tag{77}$$

$$1 \otimes \sum_{\tilde{i}=1}^{3} (\tilde{a}_{\tilde{i}}^\dagger \tilde{a}_{\tilde{i}} - \tilde{\theta}_{\tilde{i}}^\dagger \tilde{\theta}_{\tilde{i}}) \equiv \tilde{\Sigma}_3' \tag{78}$$

and

$$1 \otimes \sum_{\tilde{i}=1}^{3} (\tilde{a}_{\tilde{i}}^\dagger \tilde{a}_{\tilde{i}} + \tilde{\theta}_{\tilde{i}}^\dagger \tilde{\theta}_{\tilde{i}}) \equiv \tilde{N}. \tag{79}$$

However, $\tilde{\Sigma}_1'$, $\tilde{\Sigma}_2'$, and $\tilde{\Sigma}_3'$ do not quite obey the vector-multiplication rules (75). Two of the relations are obeyed, viz.

$$[\tilde{\Sigma}_2', \tilde{\Sigma}_3'] = 2\mathrm{i}\tilde{\Sigma}_1' \tag{80}$$

and

$$[\tilde{\Sigma}_3', \tilde{\Sigma}_1'] = 2\mathrm{i}\tilde{\Sigma}_2'. \tag{81}$$

However, the third is not;

$$[\tilde{\Sigma}_1', \tilde{\Sigma}_2'] = 2\mathrm{i}\tilde{\Sigma}_3' - 4\mathrm{i}\tilde{P} \tag{82}$$

where

$$\tilde{P} = 1 \otimes \sum_{\tilde{i},\tilde{j}=1}^{3} \tilde{\theta}_{\tilde{i}}^\dagger \tilde{\theta}_{\tilde{j}} \tilde{a}_{\tilde{j}}^\dagger \tilde{a}_{\tilde{i}}. \tag{83}$$

In addition, the $\tilde{\Sigma}_{\tilde{i}}'^2$ do not equal the unit matrix as they should in a two-dimensional representation of $SU(2)$. Rather, $\tilde{\Sigma}_1'^2 = \tilde{\Sigma}_2'^2 = \tilde{N}$ and $\tilde{\Sigma}_3'^2 \sim \tilde{N}^2$.

If, on the other hand, operators can be found which *do* obey the vector-multiplication rules of $SU(2)$ when operating on the doublets, then it may be possible to identify all three generations of quarks and leptons with representations of (12) extended to four dimensions, i.e. with

$$\tilde{H} = \sqrt{2}\tilde{\alpha} \sum_{\tilde{j}=1}^{4} (\tilde{\theta}_{\tilde{j}}^\dagger \tilde{a}_{\tilde{j}} + \tilde{\theta}_{\tilde{j}} \tilde{a}_{\tilde{j}}^\dagger). \tag{84}$$

The eigenfunctions of (84) are exact $SU(4)$ representations, and resemble those depicted in figure 2: a ground state, two **4**s in place of the **3**s, two **10**s in place of the **6**s, etc. In an

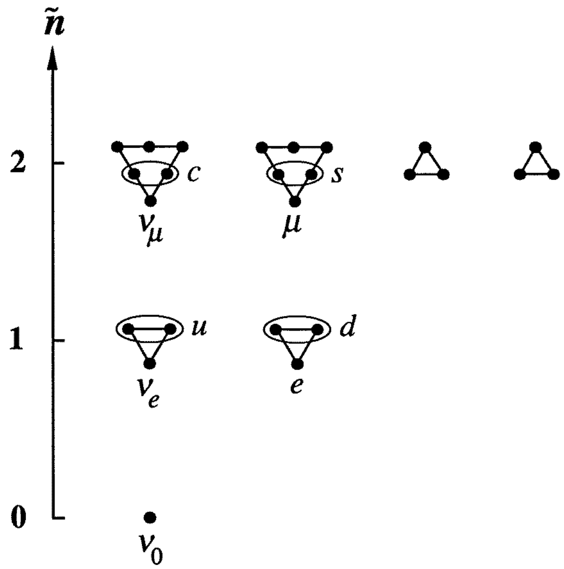

Figure 3. Weight diagrams of $\tilde{n} = 0$–2 irreps of the mass operator $\tilde{H}$; some Dirac-particle assignments are indicated.

$SU(3) \otimes U(1)$ decomposition, each **4** breaks to $\mathbf{3} \oplus \mathbf{1}$, each **10** breaks to $\mathbf{6} \oplus \mathbf{3} \oplus \mathbf{1}$, etc. Leptons would be identified with the **1**s and quarks with the **3**s.

These assignments can be inferred from the seven-dimensional model if we imagine that the $SU(3)$ states are replaced by toy $SU(2)$ states. Then quark triplets become doublets, leptons remain singlets, and the first two generations can be identified with the $\tilde{n} = 0$–2 irreps as in figure 3. The ground state could be a candidate for dark matter. Other unidentified states would be predictions.

Acknowledgments

We would like to thank Richard Arnowitt, Robert Clark, Stephen Fulling, Chris Pope, Roger Smith and Philip Yasskin for helpful discussions.

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